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Discrete Mathematics 307 (2007) 402–408
DISCRETE MATHEMATICS
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Hadwiger's conjecture for circular colorings of edge-weighted graphs ☆

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Received 2 July 2003; received in revised form 19 May 2004; accepted 26 September 2005
Available online 6 September 2006

Abstract

Let $G^w = (V, E, w)$ be a weighted graph, where $G = (V, E)$ is its underlying graph and $w : E \to [1, \infty)$ is the edge weight function. A (circular) p-coloring of G^w is a mapping c of its vertices into a circle of perimeter p so that every edge $e = uv$ satisfies $\mathrm{dist}(c(u), c(v)) \geqslant w(uv)$. The smallest p allowing a p-coloring of G^w is its circular chromatic number, $\chi_c(G^w)$.

A p-basic graph is a weighted complete graph, whose edge weights satisfy triangular inequalities, and whose optimal traveling salesman tour has length p. Weighted Hadwiger's conjecture (WHC) at $p \geqslant 1$ states that if p is the largest real number so that G^w contains some p-basic graph as a weighted minor, then $\chi_c(G^w) \leqslant p$.

We prove that WHC is true for $p < 4$ and false for $p \geqslant 4$, and also that WHC is true for series–parallel graphs.

Keywords: Edge-weighted graph; Circular coloring; Edge-weighted minor; Hadwiger's conjecture

1. Introduction and motivation

Hadwiger's conjecture (HC) states that every graph with chromatic number at least k contains the complete graph on k vertices as a minor [5]. Proving validity of HC for $k \leqslant 4$ turns out to be relatively straightforward; for $k = 5$ and 6, however, HC is equivalent to the Four Color Theorem [13,11]. HC is open for $k \geqslant 7$, and remains one of the most important and possibly most difficult problems in graph theory. We refer the reader to a recent survey article [12].

Graph colorings serve as important examples in discrete optimization. Many real-life applications, scheduling for example, turn out to be nothing more than coloring graphs with, say, as few colors as possible. Coloring problems are typically hard. If $k \geqslant 3$, deciding whether a general graph G admits a coloring with at most k colors is an NP-complete problem [4].

In this paper we shall translate HC into the language of colorings of weighted graphs. We shall prove that weighted HC (WHC) is true for series–parallel graphs, and false in general. This result may indicate that coloring edge-weighted graphs is in reality a much more difficult problem than coloring their unweighted counterparts. In the last section, we shall turn our attention to graphs with integral edge weights and examine an integral version of WHC. We shall use standard graph theoretic terminology [2], and some results on series–parallel graphs from [3].

☆ Supported in part by the Ministry of Science and Technology of Slovenia, Research Program P1-0297 and Research Project J1-6150-0101-04.
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doi:10.1016/j.disc.2005.09.032

1.1. Setting of the problem

A weighted graph $G^w = (V, E, w)$ is formed by adjoining an *edge weight function* (or just *weights*) $w : E \to [1, \infty)$ to its *underlying graph* $G = (V, E)$.

Let $\mathscr{S}_p$ denote the circle of perimeter p, and let dist_p be the *circular distance* on $\mathscr{S}_p$. A *circular p-coloring* of a weighted graph $G = (V, E, w)$ is a mapping $c : V \to \mathscr{S}_p$ so that each edge $uv \in E$ satisfies

$$\mathrm{dist}_p(c(u), c(v)) \geqslant w(uv). \tag{1}$$

Circular chromatic number of a weighted graph G^w, $\chi_c(G^w)$, is the infimum of all $p \in \mathbb{R}$, for which a p-coloring of the weighted graph G^w exists. It is shown in [9] that the infimum is attained, i.e., G^w admits a $\chi_c(G^w)$-coloring.

Observe that if all the weights are equal to 1 (in this case we identify the "weighted graph" G^w with its underlying graph G), then $\chi_c(G^w)$ is the usual circular chromatic number $\chi_c(G)$ [14], and also $\chi(G) = \lceil \chi_c(G^w) \rceil$.

A problem related to the circular chromatic number of G^w is the *span problem* (or *channel assignment problem* if G^w has integer weights): find an interval I of shortest length ℓ, so that there exists a mapping of vertices into I which satisfies the coloring condition (1). Such an ℓ is called the *span* of G^w. We refer to [8] for further details about the channel assignment problem.

There is a fundamental difference between the span problem on one hand, and traditional graph coloring and circular coloring of either unweighted or weighted graphs on the other. The space of colors in the former case is not symmetric: if we make an unfortunate choice for the color of a single vertex, we may not extend it to the coloring of the whole graph, even if the color space was large enough in the first place.

1.2. Weighted subgraphs and minors

Edges and in particular edge weights can be considered as constraints when coloring vertices of a graph. We call $H^{w'} = (V', E', w')$ a *weighted subgraph* of $G^w = (V, E, w)$ if $H \subseteq G$ (the underlying graphs must be in the usual subgraph relation) and $w'(e) \leqslant w(e)$ for every edge $e \in E'$. Hence, we construct subgraphs of $G^w = (V, E, w)$ by deleting edges, deleting vertices, and also *decreasing* edge weights. Let us stress once again that an edge weight is least 1. This also implies that if c is a p-coloring of G^w, its restriction to vertices of $H^{w'}$ is a p-coloring of $H^{w'}$.

How do we contract an edge $e = uv$ in a weighted graph G^w? The resulting graph G^w/e has G/e as its underlying graph. Its weight function, $w_{/e}$, is defined in the following way: let z be the vertex obtained by identifying vertices u and v. If an edge f is not incident with u or v, then $w_{/e}(f) = w(f)$. Similarly, if a vertex $x \in V(G^w)$ is adjacent to at most one of u or v, say u, then $w_{/e}(zx) = w(ux)$. If $x \in V(G^w)$ is adjacent to both u and w, then $w_{/e}(xz) = \max\{w(xu), w(xv)\}$.

As in the unweighted case, $H^{w'}$ is a (*weighted*) *minor* of G^w, $H^{w'} \leqslant G^w$, if it can be obtained from G^w by taking (weighted) subgraphs and contracting edges.

If G^w is a weighted graph with all weights equal to 1, then every weighted minor and every weighted subgraph of G^w has all edge weights equal to 1. The weighted minor and weighted subgraph relations are extensions of the usual subgraph and minor relations to the class of edge-weighted graphs.

1.3. Series–parallel graphs

There exist several characterizations of *series–parallel graphs*: the blocks of a series–parallel graph can be recursively constructed from a K_2 by the operations of subdividing and of doubling edges. In terms of forbidden minors, G is a series–parallel graph, if and only if G does not contain a K_4 minor. For the purposes of this paper we shall use the following characterization:

Edge-maximal series–parallel graphs with at least three vertices are constructed recursively from copies of K_3 by pasting along K_2's [3, Proposition 8.3.1]. This implies that edge-maximal series–parallel graphs are chordal, and do not contain cutvertices. General series–parallel graphs are obtained from edge-maximal ones by deleting edges.

An edge $e = uv$ *splits* G if $G - u - v$ is disconnected. We say that (G_1, G_2) is a *separation of* G *along* e if $V(G_1 \cap G_2) = \{u, v\}$, $G_1 \cup G_2 = G$, and both G_1 and G_2 are proper subgraphs of G.

If x and y are nonadjacent vertices in an edge-maximal series–parallel graph then there exists and edge $e \in E(G - u - v)$ and a separation (G_1, G_2) of G along e, so that $x \in V(G_1)$ and $y \in V(G_2)$. Consider, for example, a chord in a shortest cycle containing both x and y.

2. Results

If G is an unweighted graph with $\chi(G)=k$, then G contains a *k-critical* subgraph. Hajós' Theorem [6] states that every k-critical graph can be inductively constructed from K_k by identifying nonadjacent vertices and performing the so-called *Hajós sum* operation.

Let $h(G)$, known also as the Hadwiger number, denote the largest integer k, such that G contains a minor isomorphic to K_k. It is the complete graphs that link Hajós' Theorem with HC:

Hadwiger's conjecture (HC) (*Hadwiger [5]*). Let $k \in \mathbb{N}$ and let G be a graph. If $h(G) < k$ then $\chi(G) \leqslant h(G)$.

Note that condition $h(G) < k$ is equivalent to $h(G) \leqslant k-1$.

Theorem 1 (*Hadwiger [5], Wagner [13], Appel and Haken [1], Robertson et al. [11]*). *Hadwiger's Conjecture is true for $k \leqslant 6$.*

Mohar has proven a version of Hajós' Theorem for circular colorings of edge-weighted graphs [10, Theorem 6.4]: a *p-basic* graph is a weighted complete graph, whose weights satisfy triangular inequalities, and whose optimal traveling salesman tour has length p. Every p-critical weighted graph (all its proper subgraphs have strictly smaller circular chromatic numbers) can be in the limit sense constructed by identifying nonadjacent vertices and using *strong Hajós sums* starting from p-basic graphs, see [10].

In the weighted version of Hajós' Theorem p-basic graphs play the role of complete graphs. We therefore use p-basic graphs in place of complete graphs also in the weighted version of HC. As an analogue of the above-mentioned Hadwiger number, the *weighted Hadwiger number of G^w*, $h'(G^w)$, is the largest p, so that G^w contains a p-basic minor.

Observe that for $q > p$, a q-basic graph may contain no p-basic graph as a minor. For example, a K_4 with all weights equal to 1 does not contain any $\frac{7}{2}$-basic graph as a minor.

Weighted Hadwiger's conjecture (WHC). If G^w is an edge-weighted graph satisfying $h'(G^w) = p$ for some real number $p \geqslant 1$, then $\chi_c(G^w) \leqslant p$.

Consider a weighted graph G^w, whose edge weights are all equal to 1, and let G be its underlying graph. Now, $h'(G^w) = h(G) \in \mathbb{N}$, $\chi_c(G^w) = \chi_c(G)$, and $\chi(G) = \lceil \chi_c(G) \rceil$. This implies that WHC restricted to the class of graphs with unit edge weights is equivalent to the standard HC.

However, WHC is false in general. Let us construct a graph G taking a copy of K_3 (which we shall call the central K_3) with vertex set $\{v_1, v_2, v_2\}$ and three copies of K_4's. Identify the three edges of the central K_3 with three edges from distinct copies of K_4. Observe that G is a perfect graph. Its (unweighted) chromatic number $\chi(G)$ equals its clique size, namely 4.

Let G^w be the weighted graph whose underlying graph is G, and whose weights are equal to 1, except for the edges on the central K_3, whose edge weights are all equal $\omega \in (1, \frac{7}{6})$. It is easy to see that $h'(G^w) = 4$.

We can extend the mapping $v_1 \mapsto 0$, $v_2 \mapsto \omega$, $v_3 \mapsto 2\omega$ to a $(2+2\omega)$-coloring of G^w. This shows that $\chi_c(G^w) \leqslant 2 + 2\omega$.

Assume that G^w admits a p-coloring c where $p < 2 + 2\omega$. As K_4^1 is a subgraph of G^w, $p \geqslant 4$. Since $2\omega > p/2$ it follows that

$$\mathrm{dist}_p(c(v_1), c(v_2)) + \mathrm{dist}_p(c(v_2), c(v_3)) + \mathrm{dist}_p(c(v_3), c(v_1)) = p. \tag{2}$$

As the coloring of the central K_3 may be extended to K_4's we infer that for $1 \leqslant i < j \leqslant 3$ either

$$2 \leqslant \mathrm{dist}_p(c(v_i), c(v_j)) \leqslant p/2 \quad \text{or} \tag{3}$$

$$\omega \leqslant \mathrm{dist}_p(c(v_i), c(v_j)) \leqslant p - 3. \tag{4}$$

It follows directly from (2) that at least one edge $v_i v_j$ in central K_3 satisfies (3) and also at least one edge satisfies (4). Therefore, we obtain

$$\mathrm{dist}_p(c(v_1), c(v_2)) + \mathrm{dist}_p(c(v_2), c(v_3)) + \mathrm{dist}_p(c(v_3), c(v_1)) \geqslant 2 + 2\omega > p,$$

which is a contradiction. Hence, $\chi_c(G^w) = 2 + 2\omega > 4 = h'(G^w)$.

By multiplying all weights of G^w with a constant $q \geqslant 1$ we obtain a graph G^{qw}, which satisfies the relation $h'(G^{qw}) = 4q < \chi_c(G^{qw})$.

Theorem 2. *WHC is true for series–parallel graphs and, in particular, it is true for $p < 4$: if G^w is a weighted series–parallel graph or $h'(G^w) \leqslant p < 4$, then $\chi_c(G^w) \leqslant h'(G^w)$.*

We shall devote the rest of this section to the proof of Theorem 2, and start with a straightforward yet important lemma. We need the following definitions.

Let c be a p-coloring of G^w. The *c-span* of an edge e, $c(e)$, is defined as $\mathrm{dist}_p(c(u), c(v))$. Clearly, $c(e) \leqslant p/2$. We also say that an edge e is *c-tight* (or just *tight*) if $c(e) = w(e)$.

Lemma 3. *Let $e = uv$ be an edge which splits G^w, and let (G_1^w, G_2^w) be the weighted separation of G^w along e. Let c_1 be a p_1-coloring of G_1^w, so that e is c_1-tight in G_1^w, and let c_2 be a p_2-coloring of G_2^w, so that e is c_2-tight also in G_2^w. Let $e_1 \neq e$ be an arbitrary edge of G_1^w, let $e_2 \neq e$ be an arbitrary edge of G_2^w, and let $p \geqslant \max\{p_1, p_2\}$.*

Then we can combine colorings c_1 and c_2 to a p-coloring c of the whole graph G^w, so that e is c-tight in G^w, $c(e_1) = c_1(e_1)$, and $c(e_2) = c_2(e_2)$.

Proof. By rotating and/or reflecting the colors in each of c_1 and c_2 we may assume that $c_1(v) = c_2(v) = 0$, and $c_1(v) = c_2(u) = w(e)$. We may view edges of G_i^w as being mapped on the short arcs between colors of their endvertices in $\mathscr{S}_{p_i}$, $i = 1, 2$. In particular c_i, $i = 1, 2$, maps the edge e on the arc $[0, w(e)]$ between 0 and $w(e)$ in $\mathscr{S}_i$.

Let $e_i = u_i v_i$, $i = 1, 2$. If $\{c_i(u_i), c_i(v_i)\} \neq \{0, w(e)\}$ then $[0, w(e)]$ and a short arc between $c_i(v_i)$ and $c_i(u_i)$ do not cover $\mathscr{S}_{p_i}$. If $\{c_i(u_i), c_i(v_i)\} = \{0, w(e)\}$ then $[0, w(e)]$ is also a short arc between $c_i(v_i)$ and $c_i(u_i)$. In any case we may find a point $x_i \in \mathscr{S}_{p_i} \backslash [0, w(e)]$ which lies off some short arc between $c_i(v_i)$ and $c_i(u_i)$, $i = 1, 2$.

For $i = 1, 2$ let us define mappings $\overline{c_i} : V(G_i^w) \to \mathscr{S}_{p_i}$ with

$$\overline{c_i}(v) = \begin{cases} c_i(v) & \text{if } c_i(v) \in [0, x_i], \\ c_i(v) + p - p_i & \text{otherwise.} \end{cases}$$

The union $\overline{c_1} \cup \overline{c_2}$ is a p-coloring of G^w with the desired properties. $\square$

If a weighted graph G^w satisfies $h'(G^w) < 4$, then its underlying graph G does not contain a K_4 minor. In other words, $h'(G^w) < 4$ implies that G^w is a weighted series–parallel graph. As every series–parallel graph is obtained as a subgraph of an edge-maximal series–parallel graph, it is enough to prove Theorem 2 for edge-maximal series–parallel graphs on at least three vertices.

Let $h_3(G^w)$ be the maximal $w(e_1) + w(e_2) + w(e_3)$ taken over the set of all triplets of edges $\{e_1, e_2, e_3\}$ lying on a common cycle. Similarly, let $h_2(G^w) = 2\max_{e \in E(G)}\{w(e)\}$. As G^w is series–parallel and does not contain a K_4 minor, $h'(G^w) = \max\{h_2(G^w), h_3(G^w)\}$.

Choose an arbitrary edge $e_a \in G^w$ with maximal edge weight. We say that e_a is the *α-edge* in G^w. Next, let $e_b (\neq e_a)$ be the edge satisfying $w(e_b) \geqslant w(e)$ for all $e \in E(G^w - e_a)$. We say that e_b is the *β-edge* in G^w. Obviously, $w(e_a) \geqslant w(e_b)$ and it may happen that $w(e_a) = w(e_b)$.

Let R be the union of cycles passing through both e_a and e_b in G^w. Since G^w is a 2-connected graph R is not empty. Choose any edge e_c that has maximal edge weight in $R - e_a - e_b$. We call e_c the *γ-edge* of G^w.

The following proposition constructs a p-coloring of G^w which serves as the last step in the proof of Theorem 2.

Proposition 4. *Let G^w be a 2-connected edge-maximal series–parallel graph, e_a its α-edge, e_b its β-edge, e_c its γ-edge, and let $p = h'(G^w)$.*

(a) *If $w(e_b) \geqslant \frac{1}{2}w(e_a)$ then there exists a p-coloring c of G^w so that both e_b and e_a are c-tight.*
(b) *If $w(e_b) < \frac{1}{2}w(e_a)$ then there exists a p-coloring c of G^w so that e_a is c-tight.*

Proof. If $\frac{1}{2}w(e_a) > w(e_b)$ then $h'(G^w) = h_2(G^w) = 2w(e_a)$. We increase the weight of e_b to $h'(G^w) - w(e_a) - w(e_c)$, and we do not alter the weighted Hadwiger number of G^w by doing so.

Hence, we may assume that $w(e_b) \geqslant \frac{1}{2} w(e_a)$. Now we replace the weight of e_c with $h'(G^w) - w(e_a) - w(e_b)$. Since $w(e_b)$ was large enough the new weight of e_c is not larger that $w(e_b)$, and as above $h'(G^w)$ is left unchanged.

We shall henceforth assume that $w(e_a) + w(e_b) + w(e_c) = h'(G^w)$. If $|V(G^w)| = 3$ then we can p-color G^w so that all three edges e_a, e_b, and e_c are tight.

The inductive step on $|V(G^w)|$ is treated in several cases.

(a) *The edge e_b splits G^w.*

Let (G_1^w, G_2^w) be the separation of G^w along e_b. Observe that e_b is an α-edge in, say, G_1^w and a β-edge in G_2^w. Both G_1^w and G_2^w are minors in G^w, so $p_i := h'(G_i^w) \leqslant h'(G^w) = p$, for $i = 1, 2$. Let c_1 be an inductively obtained p_1-coloring of G_1^w in which e_a is tight, and let c_2 be the p_2-coloring of G_2^w in which both e_a and e_b are tight. Finally, we combine c_1 and c_2 according to Lemma 3 so that e_b stays tight in the combined coloring.

(b) *The edge e_a splits G^w.*

This case is treated in a similar way as case (a).

(c) *Edges e_a and e_b do not lie on a common triangle.*

In this case let $e \neq e_a, e_b$ be an edge of G^w so that for some separation (G_1^w, G_2^w) along e we have $e_a \in E(G_1^w)$ and $e_b \in E(G_2^w)$. Observe, that e_a is the α-edge in G_1^w, and that e_b can be chosen as the α-edge in G_2^w.

Next, we replace the weight of e in both G_1^w and G_2^w with $w(e_b)$. (As e_b was an edge with second largest weight in G^w, the weight of e is not decreased by this operation.) Now e is an edge with the second largest weight in both G_1^w and G_2^w and can be chosen as the β-edge in either of these graphs. Observe that G_1^w and G_2^w equipped with the new weights are both minors of G^w. Inductively, we find colorings c_1 and c_2, so that e_a and e are c_1-tight in G_1^w, so that e_b and e are c_2-tight in G_2^w, and combine c_1 and c_2 according to Lemma 3.

(d) *None of e_a, e_b splits G^w, and e_a,e_b lie on a common triangle t.*

Since e_a and e_b do not split G^w but do lie on a common triangle, every edge $e \in E(G^w - e_a - e_b)$ lies on cycle passing through both e_a and e_b. By the definition of the γ-edge, $w(e) \leqslant w(e_c)$ for every edge $e \in E(G^w - e_a - e_b)$. We replace the weight of every edge, apart from e_a and e_b, with $w(e_c)$, and do not change $h'(G^w)$ by doing so.

Now let e_t be the third edge in t. Let (G_1^w, G_2^w) be the separation of G^w along e_t, so that G_1^w contains exactly three edges e_a, e_b, and e_t.

As every edge in G_2^w has the same weight $w(e_c)$ we $(3w(e_c))$-color G_2^w as an unweighted graph, so that all edges are tight. The graph G_1^w consists of a single triangle, so let c_1 be a p-coloring of G_1^w, in which e_a, e_b, and e_t are all c_1-tight. Clearly, we may combine the colorings into a desired p-coloring of G^w.

Cases (a)–(d) treat all possibilities when $|V(G^w)| \geqslant 4$, and the proof is complete. $\square$

3. Integral colorings

In this section we shall turn our attention to the integral version of WHC. A weighted graph G^w is called *integral* if the weights in G^w are integers. Let $h_i'(G^w)$ denote the largest p, so that G^w contains some integral p-basic graph as a minor.

Lemma 5. *Let G^w be an integral weighted graph. Then*

(a) *$H^{w'} \leqslant G^w$ implies that also $H^{\lceil w' \rceil} \leqslant G^w$, and*
(b) *$h_i'(G^w) = h'(G^w)$.*

Proof. We leave the proof of (a) to the reader and focus on (b). By definition $h_i'(G^w) \leqslant h'(G^w)$. In order to prove the reverse inequality let $p = h'(G^w)$ and let $H^{w'}$ be a p-basic minor of G^w. By (a) $H^{\lceil w' \rceil} \leqslant G^w$, and since $a + b \geqslant c$ implies $\lceil a \rceil + \lceil b \rceil \geqslant \lceil c \rceil$, the weights of $H^{\lceil w' \rceil}$ satisfy triangle inequalities. Hence, also $p \leqslant h_i'(G^w)$. $\square$

Let k be an integer. An *integral circular k-coloring* of an integral weighted graph $G^w = (V, E, w)$ is a mapping $c : V \to \mathscr{S}_k$, which is a circular k-coloring (in the usual weighted sense) and whose image is contained in $\mathbb{Z}_k$ (which we may view as a subset of $\mathscr{S}_k$). We shall use $\chi_c^i(G^w)$ to denote the integral circular chromatic number of G^w.

We can now formulate the integral WHC (IWHC):

Integral weighted Hadwiger's conjecture (IWHC). Let $k \in \mathbb{N}$ and let G^w be an integral weighted graph. If $h_i'(G^w) < k$, then $\chi_c^i(G^w) \leqslant h_i'(G^w)$.

Theorem 6. *IWHC is valid for* $k \leqslant 6$.

Proof. Let $\mathscr{I}_k$ denote the set of (weighted) minor-minimal integral graphs in

$$\{H^w \mid H^w \text{ is weighted, integral, and } k'\text{-basic for some } k' \geqslant k\}.$$

If G^w is an integral weighted graph, then $h_i'(G^w) < k$ if and only if G^w does not contain a weighted minor from $\mathscr{I}_k$.

The reader will have no difficulties showing that $\mathscr{I}_2 = \{K_2^1\}$ (where K_2^1 is a graph whose weight function is a constant 1), $\mathscr{I}_3 = \{K_2^2, K_3^1\}$, and $\mathscr{I}_4 = \{K_2^2, K_4^1\}$. If $h_i'(G^w) < 2$ then G^w has no edges, and if $h_i'(G^w) < 3$, then G^w is a tree and all its weights are equal to 1. If $h_i'(G^w) < 4$, then G^w is a series–parallel graph and again its weights are equal to 1. These cases reduce to Theorem 1.

A little more careful analysis shows that $\mathscr{I}_5 = \{K_2^3, K_3^{(2,2,1)}, K_5^1\}$, where the edge weights of the three edges in $K_3^{(2,2,1)}$ are 1, 2, and 2, respectively. Similarly, $\mathscr{I}_6 = \{K_2^3, K_3^2, K_4^W, K_6^1\}$, where W is a 1–2 weight function, giving weight 2 to exactly three edges sharing a common vertex.

We shall leave the proof of case $k = 5$ to the reader and proceed with the proof in case $k = 6$: let G^w be a counterexample which has minimum number of vertices and maximum possible sum of edge weights. Hence, $h_i'(G^w) \leqslant 5$ and $\chi_{wi}(G^w) \geqslant 6$. As $\mathscr{I}_6$ contains K_2^3 the weight of every edge in G^w is either 1 or 2.

Consider the underlying graph G of G^w. As $h_i'(G^w) \leqslant 5$ also $h(G) \leqslant 5$. By Theorem 1, G admits a 5-coloring c (with colors from $\mathbb{Z}_5$) which fails to be a 5-coloring of G^w.

If fewer than three edges of G^w have weight 2, we can adjust c to a coloring of G^w by permuting colors. If G^w is not 2-connected then partial colorings of its blocks may be combined to give a coloring of G^w. Hence, at least three edges of G^w have weight 2 and G^w is 2-connected.

Suppose that deleting both vertices u and v disconnects G^w and let (G_1^w, G_2^w) be the corresponding separation. By maximality of sum of weights, uv is an edge of G^w. If $w(uv) = 2$ the 5-colorings of G_1^w and G_2^w may be combined (by rotating and/or reflecting colors) to give an appropriate 5-coloring of G^w. Hence, $w(uv) = 1$.

Assume that all edges of G_1^w have weight equal to 1. We can 5-color G_2^w by induction. Next, we color G_1 as an unweighted graph, and combine the two colorings to give a 5-coloring of G^w by permuting colors in G_1^w *only*. So we may assume that both G_1^w and G_2^w contain an edge with weight 2. Denote by $G_i^{w,2}$ the graph G_i^w with the weight of uv replaced with 2. Both $G_1^{w,2}$ and $G_2^{w,2}$ are weighted minors of G^w and can 5-colored inductively. These colorings may be combined (by rotating and/or reflecting colors only) to a 5-coloring of G^w.

Finally, we assume that G^w is 3-connected. Let e_1, e_2, e_3 be three edges with weight 2. As K_3^2 is not a weighted minor of G^w, the edges e_1, e_2, and e_3 do not lie on a common cycle. By [7, Exercise 6.67] this is only possible if they form a 3-edge cut or share a common vertex u. In the latter case let u_i be the other endvertex of e_i $(i = 1, 2, 3)$. Recall that $G - u$ is 2-connected. We may either find a cycle in $G - u$ through all u_i's or a cycle C containing u_1 and u_2 together with a u_3–$(C - u_1 - u_2)$ path which avoids u. An appropriate contraction yields a K_4^W. The 3-edge cut case can be settled similarly. □

Acknowledgment

The author's research was conducted while visiting University of Hannover under sponsorship of Alexander von Humboldt Foundation. Both hospitality of the university and help of the foundation are greatly acknowledged.

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